THE LIMITING DISTRIBUTION OF CHARACTER SUMS

AYESHA HUSSAIN

ABSTRACT. In this paper, we consider the distribution of the continuous paths of Dirichlet character sums modulo prime q on the complex plane. We also find a limiting distribution as $q \to \infty$ using Steinhaus random multiplicative functions, stating properties of this random process. This is motivated by Kowalski and Sawin's work on Kloosterman paths.

1. INTRODUCTION

Given a primitive Dirichlet character χ modulo q, we define the normalised partial character sum

$$S_\chi(t) := \frac{1}{\sqrt{q}} \sum_{n \leq qt} \chi(n),$$

for $t \in [0, 1]$. Such character sums play a fundamental role in analytic number theory. Our goal is to study the distribution of character sums for prime modulus q, and find the limiting distribution as $q \to \infty$, answering the open problem set by Kowalski and Sawin [13, Section 5.2].

When investigating the maximum of these character sums, Bober, Goldmakher, Granville and Koukoulopoulos [2] studied the distribution function for $\tau > 0$,

$$(1) \qquad \Phi_q(\tau) := \frac{1}{\phi(q)} \# \left\{ \chi \mod q : \max_t |S_\chi(t)| > \frac{e^\gamma}{\pi} \tau \right\}.$$

The limiting distribution of Φ_q is

$$\Phi(\tau) := \mathbb{P}\left(\max_t |F(t)| > 2e^\gamma \tau \right),$$

where $F(t)$ is a random Fourier series defined later in this paper [2, Theorem 1.4]. We find, through different methods, that the limiting distribution of character sums, not just their maxima, uses the same random series. Our main theorem can also be used to recover their result.

The character sum $S_\chi(t)$ is a step function, with jump discontinuities at every $t \in \frac{1}{q}\mathbb{Z}$. In order to circumvent the difficulties posed by these discontinuities, it is natural to consider a continuous modification, where the steps are replaced by straight line interpolations.
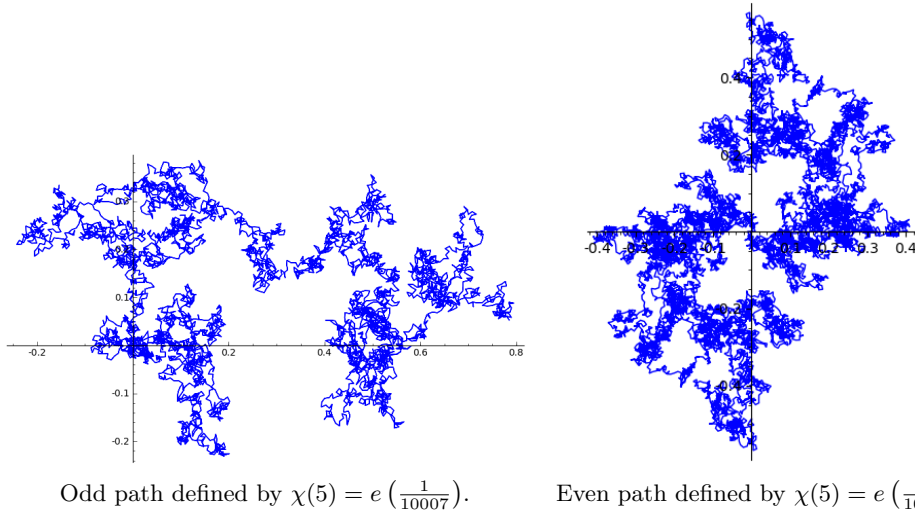
Definition 1.1. *Character paths are paths in the complex plane formed by drawing a straight line between the successive partial sums*

$$S_\chi(x) = \tfrac{1}{\sqrt{q}} \sum_{n \leq qx} \chi(n), \quad S_\chi(x + 1/q) = \tfrac{1}{\sqrt{q}} \sum_{n \leq qx+1} \chi(n),$$

for $x \in [0, 1)$ and $qx \in \mathbb{Z}$. We parameterise character paths by the function

$$f_\chi(t) := S_\chi(t) + \frac{\{qt\}}{\sqrt{q}} \chi(\lceil qt \rceil),$$

where $\{x\}$ is the fractional part of the number x.

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Odd path defined by $\chi(5) = e\left(\frac{1}{10007}\right)$.   Even path defined by $\chi(5) = e\left(\frac{2}{10007}\right)$.

FIGURE 1. Character paths of an odd and even character modulo 10007.

Note the difference between character sums and character paths is bounded by $\frac{1}{\sqrt{q}}$. Character paths, like character sums, are periodic and have the truncated Fourier series

$$(2) \qquad \frac{\tau(\chi)}{2\pi i \sqrt{q}} \sum_{0<|k|<q} \frac{\overline{\chi}(k)}{k}(1 - e(-kt)) + O\left(\frac{\log q}{\sqrt{q}}\right).$$

Character paths are polygonal, continuous and closed. Examples of character paths can be seen in Figure 1.

We define the distribution of character paths by taking $\chi \bmod q \mapsto f_\chi(t)$ as a random process, choosing χ uniformly at random. Let us write $\mathcal{F}_q$ for this random process,

$$\mathcal{F}_q(t) := \{f_\chi(t) : \chi \bmod q\}.$$

One of the main goals of this paper is to find the limiting distribution of the sequence $(\mathcal{F}_q)_q$ as $q \to \infty$ through the primes. For this, we need Steinhaus random multiplicative functions.

Definition 1.2. (1) Steinhaus random variables X_p are random variables uniformly distributed on the unit circle $\{|z| = 1\}$.

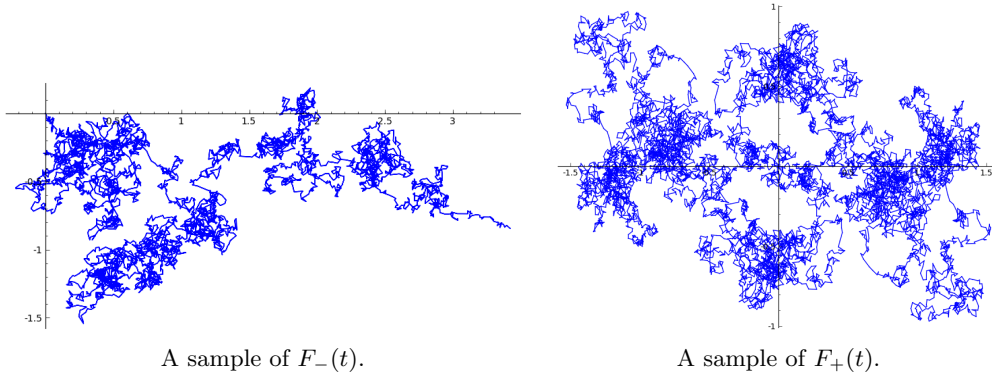(2) Steinhaus random multiplicative functions X_n, $n \in \mathbb{N}$, are defined as

$$X_n = \prod_{p^a \| n} X_p^a,$$

where X_p are Steinhaus random variables for prime p. We extend this definition to $n \in \mathbb{Z}$ by taking $X_{-1} = \pm 1$ with probability $1/2$ each, so $X_{-n} = X_{-1}X_n$. (Here $p^a \| n$ means p^a strictly divides n, so $p^a | n$ but $p^{a+1} \nmid n$).

Steinhaus random multiplicative functions are completely multiplicative, with all values distributed on the unit circle. This leads to a natural question: can we compare partial sums of characters with partial sums of Steinhaus random multiplicative functions, assuming similar behaviour? Sums of Steinhaus random multiplicative functions might be a good model for short character sums, but the periodicity of the characters means that for long character sums the model fails. Instead, we consider X_n as Fourier coefficients. The sums of Steinhaus random multiplicative functions have a long history. See Harper [10] for an example of recent work on this.

Consequently, we must find a way to incorporate the periodicity from the character sums. Let $F(t)$ be the random Fourier series

$$(3) \qquad F(t) := \frac{\eta}{2\pi} \sum_{|k|>0} \frac{X_k}{k}(1 - e(kt)),$$

A sample of $F_-(t)$.

A sample of $F_+(t)$.

FIGURE 2. Samples of $F_\pm$ with 10007 terms.

where X_k are Steinhaus random multiplicative functions for $k \neq 0$ and η is a random variable uniformly distributed on the unit circle. Additionally, take $F_\pm(t)$ where we fix X_{-1} as $+1$ or -1. The infinite series is well defined, and we show in Section 2 that this is almost surely the Fourier series of a continuous function. Therefore we can think of F as a random process on $C([0, 1])$. Examples of the paths formed by $F_\pm$ are shown in Figure 2.

Using the random Fourier series F, we state the main theorem of the paper:

Theorem 1.1. *Let $F_\pm$ be defined as above for $t \in [0, 1]$. The sequence of the distributions of character paths $(\mathcal{F}_{q,\pm}(t))_q$ weakly converges to the process $F_\pm(t)$ in the Banach space $C([0, 1])$ as $q \to \infty$ through the primes. In other words, for any continuous and bounded map*

$$\psi : C([0, 1]) \to \mathbb{C},$$

we have, for prime q,

$$\lim_{q \to \infty} \mathbb{E}\left(\psi\left(\mathcal{F}_{q,\pm}\right)\right) = \mathbb{E}\left(\psi(F_\pm)\right).$$

1.1. **Proof Outline.** Theorem 1.1 shows that Steinhaus random multiplicative functions can be used as Fourier coefficients of a random process F to find the limiting distribution of character paths. In Section 2 we properly define the random process F and prove some of its properties. Theorem 1.1 only makes sense if $F(t)$ is a function almost surely in $C([0, 1])$, which we prove in Theorem 2.2.

The proof of Theorem 1.1 can be split into two parts: proving that the sequence $(\mathcal{F}_{q,\pm}(t))_q$ converges in finite distributions to the random process $F_\pm(t)$ and that the sequence of distributions is relatively compact. Convergence in finite distributions is proved in Section 3, using the method of moments. To prove relative compactness, it is sufficient to use Prohorov's Theorem [1, Theorem 5.1], which states that if a family of probability measures is tight, then it must be relatively compact. Section 4 proves the sequence of distributions $(\mathcal{F}_{q,\pm}(t))_q$ satisfies the tightness property, therefore proving Theorem 1.1.

Remark 1.1. As referred to earlier, Bober, Goldmakher, Granville and Koukoulopoulos [2] investigated the distribution function $\Phi_q(\tau)$, defined in Equation (1). They proved $(\Phi_{q,\pm}(\tau))$ converges weakly to

$$\mathbf{Prob}\left(\max_t |F_\pm(t)| > 2e^\gamma \tau\right).$$

as $q \to \infty$ through the primes. Theorem 1.1 can be used to obtain the same result.

Remark 1.2. The definition of character paths is motivated by similar research by Kowalski and Sawin [13, 12]. In their papers they define *Kloosterman paths*, $K_p(t)$, view the paths as random variables, and find their limiting distribution as $p \to \infty$. My work continues in this vein to investigate the analogous limiting distribution of character paths. Due to the multiplicativity of Dirichlet characters, our random multiplicative coefficients X_n aren't independent. This leads to interesting properties shown in Section 2.

Remark 1.3. Theorem 1.1 is restricted to q being prime, so a natural question is to consider composite q as well. Steinhaus random multiplicative functions are non zero so we need a high percentage of primitive characters modulo q. If we take q not being too smooth we might be able to relax this condition, as the contribution from imprimitive characters could potentially be included in the error terms already produced from the method. Future work could explore the generalised case when the modulus of the characters is not prime.

Notation. We follow the usual convention of $e(x) = e^{2\pi i x}$ for all $x \in \mathbb{R}$ and $p^k \| n$ to be when $p^j | n$ for all $1 \le j \le k$ and $p^{k+1} n$. We take $d_N(x)$ as the Nth divisor function for $x \in \mathbb{N}$,

$$d_N(x) = \sum_{x_1 \cdots x_N = x} 1.$$

Given a positive integer n, we define $P^+(n)$ and $P^-(n)$ as the largest and smallest prime divisors of n. We take $P^+(1) = 1$ and $P^-(1) = \infty$ as in conventional notation.

2. PROPERTIES OF $F(t)$

Recall the random process F, defined by the infinite sum

$$F(t) = \frac{\eta}{\pi} \sum_{n \ne 0} \frac{1 - e(nt)}{n} X_n,$$

where X_n are Steinhaus random multiplicative functions, defined in Definition 1.2, and η is a random variable uniformly distributed on the unit circle. We define the infinite sum as a limit of the smooth sum

$$\frac{\eta}{\pi} \sum_{\substack{n \ne 0 \\ P^+(|n|) \le y}} \frac{1 - e(nt)}{n} X_n,$$

as $y \to \infty$. In this section we deal with samples of the random process and prove some of their properties. We will be using sample functions of the random process, which we define here for ease of notation.

Definition 2.1. *We say G is a sample function of the random process F, where*

$$G(t) := \frac{c}{\pi} \sum_{n \ne 0} \frac{1 - e(nt)}{n} \alpha_n,$$

for c a sample of a Steinhaus random variable and α_n is a sample of a Steinhaus random multiplicative function.

The Fourier coefficients are bounded by $O(1/n)$. This will be useful later in the section, where we show the infinite series F can also be defined as the limit of partial symmetric sums.

Our main result of Section 2 is proving any sample function of F is almost surely continuous. This is non trivial and involves considering the y-smooth and "y-rough" parts of the infinite sum $G(t)$.

Let

$$S_y := \sum_{\substack{n \ne 0 \\ P^+(|n|) \le y}} \frac{1 - e(nt)}{n} \alpha_n \qquad \text{and} \qquad R_y := \sum_{\substack{n \ne 0 \\ P^+(|n|) > y}} \frac{1 - e(nt)}{n} \alpha_n,$$

where $P^+(n)$ is the largest prime factor of n. Note that $S_y + R_y = \frac{\pi}{c} G(t)$ and the two functions are not independent.

Lemma 2.1. *For all $\epsilon > 0$ and sufficiently large $y > 1$,*

$$\mathbb{P}\left(\|R_y\|_\infty > \epsilon\right) \ll \exp\left\{\frac{-\epsilon^2 y^{1/3}}{\log y(\log y + O(1))}\left(\log\left(\frac{(\log y)^{20}}{\log y + O(1)}\right) + O\left(\log \epsilon\right)\right)\right\}$$

independently of S_y, where $\|\cdot\|_\infty := \max_{t\in[0,1]}|\cdot|$. Notably for all $\epsilon > 0$, we have $\mathbb{P}\left(\|R_y\|_\infty > \epsilon\right) \to 0$ as $y \to \infty$.

Proof. For all $y \geq 1$,

$$\sum_{\substack{n \geq 1 \\ P^+(n) > y}} \frac{1 - e(nt)}{n}\alpha_n = \sum_{\substack{n \geq 1 \\ P^+(n) \leq y}} \frac{\alpha_n}{n} \sum_{\substack{m > y \\ P^-(m) > y}} \frac{1 - e(mnt)}{m}\alpha_m,$$

where $P^-(m)$ is the smallest prime factor of m. By setting

$$T(\alpha) := \max_{t\in[0,1]}\left|\sum_{\substack{m > y \\ P^-(m) > y}} \frac{1 - e(mt)}{m}\alpha_m\right|,$$

we have

$$\|R_y\|_\infty := \max_{t\in[0,1]}\left|\sum_{\substack{n \neq 0 \\ P^+(n) > y}} \frac{1 - e(nt)}{n}\alpha_n\right| \leq 2 \sum_{\substack{n \geq 1 \\ P^+(n) \leq y}} \frac{T(\alpha)}{n}.$$

We then use the following result [14, Theorem 2.7],

$$\sum_{P+(n) \leq y} \frac{1}{n} = e^\gamma \log y + O(1),$$

where we took the limit of the Lemma. Consequently, $\|R_y\|_\infty$ is bounded above by

$$\|R_y\|_\infty \leq 2T(\alpha)\left(e^\gamma \log y + O(1)\right).$$

Adapting [2, Proposition 5.2] for Steinhaus random multiplicative functions[1], for $k \geq 3$ and $y \geq k^3$,

$$\mathbb{E}\left[\left(\sum_{\substack{n > y \\ P^-(n) > y}} \frac{1 - e(mt)}{m}\alpha_m\right)^{2k}\right] \ll \frac{1}{(\log y)^{40k}}.$$

We set ρ_y as the probability $T(\alpha) > \epsilon(y)$ and

$$k = \left\lfloor \frac{\epsilon(y)^2 y^{1/3}}{\log y}\right\rfloor.$$

Therefore,

$$\rho_y \leq \frac{\mathbb{E}(T(\alpha)^{2k})}{\epsilon(y)^{2k}} \ll \frac{\epsilon(y)^{-2k}}{(\log y)^{40k}} \leq \left(\frac{\epsilon(y)^{-1}}{(\log y)^{20}}\right)^{\frac{2\epsilon(y)^2 y^{1/3}}{\log y}}.$$

Taking $\epsilon(y) = \frac{\epsilon}{2e^\gamma \log y + O(1)}$ for $\epsilon > 0$,

$$\mathbb{P}\left(\|R_y\|_\infty > \epsilon\right) \leq \mathbb{P}\left(T(\alpha) > \frac{\epsilon}{2e^\gamma \log y + O(1)}\right) \ll \left(\frac{2e^\gamma \log y + O(1)}{\epsilon(\log y)^{20}}\right)^{\frac{2\epsilon^2 y^{1/3}}{\log y(2e^\gamma \log y + O(1))}}$$

$$\ll \exp\left(\frac{-\epsilon^2 y^{1/3}}{\log y(\log y + O(1))}\left(\log\left(\frac{(\log y)^{20}}{\log y + O(1)}\right) + \log\left(O(\epsilon)\right)\right)\right).$$

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[1]We take $q \to \infty$ and apply a sample of a Steinhaus random multiplicative function α_n instead of $\chi(n)$.

To prove the final part of the lemma, we take $y \to \infty$ to show the probability tends to 0,

$$0 \leq \mathbb{P}\left(\|R_y\|_\infty > \epsilon\right) \ll \lim_{y \to \infty} \exp\left(\frac{-\epsilon^2 y^{1/3}}{\log y(\log y + O(1))}\left(\log\left(\frac{(\log y)^{20}}{\log y + O(1)}\right) + O(\log \epsilon)\right)\right) = 0.$$

$\square$

Lemma 2 can be appreciated more by taking $\epsilon = 1/\log y$, leading to the following example. For sufficiently large $y > 1$,

$$\mathbb{P}\left(\|R_y\|_\infty > \frac{1}{\log y}\right) \ll \exp\left\{\frac{-y^{1/3}}{(\log y)^3(\log y + O(1))}\left(\log\left(\frac{(\log y)^{20}}{\log y + O(1)}\right) + O(\log \log y)\right)\right\}$$

independently of S_y.

Subsequently, defining F as the limit of its smooth parts, we get the following theorem.

Theorem 2.2. *Let F be the random Fourier series*

$$F(t) := \lim_{y \to \infty} \frac{\eta}{\pi} \sum_{\substack{n \neq 0 \\ P^+(|n|) \leq y}} \frac{1 - e(nt)}{n} X_n,$$

where X_n are Steinhaus random multiplicative functions and η is a random variable uniformly distributed on $\{|z| = 1\}$. With probability 1 this is the Fourier series of a continuous function.

Proof. We will prove this theorem by showing any sample of the random process F is almost surely continuous.

Consider the sequence of functions $(S_y)_y$ and $(R_y)_y$ defined by

$$S_y(t) := \frac{c}{\pi} \sum_{\substack{n \neq 0 \\ P^+(|n|) \leq y}} \frac{1 - e(nt)}{n} \alpha_n,$$

$$R_y(t) := \frac{c}{\pi} \sum_{\substack{n \neq 0 \\ P^+(|n|) > y}} \frac{1 - e(nt)}{n} \alpha_n,$$

where c, α_n are on the unit circle and $\{\alpha_n\}$ are completely multiplicative. Note that samples of the random process F can be written as $S_y(t) + R_y(t)$ for appropriate choices of α_n and c.

The function S_y is the $y-$smooth part of a sample of the random process F, which we call $G(t)$. The sum S_y converges absolutely, so $S_y(t)$ is a continuous function.

To prove continuity with probability 1, we use the first Borel-Cantelli Lemma [9, Chapter 2, Theorem 18.1] to show the $y-$rough part of G vanishes as $y \to \infty$. Consider the sequence $\{R_y : \|R_y\|_\infty > \epsilon\}_y$, where $\|R_y\|_\infty$ is the maximum of the $y-$rough part of G. Using Example 2,

$$\sum_{y=1}^\infty \mathbb{P}(\|R_y\|_\infty > \epsilon) \ll \sum_{y=1}^\infty \exp\left\{\frac{-y^{1/3}}{(\log y)^3(\log y + O(1))}\left(\log\left(\frac{(\log y)^{20}}{\log y + O(1)}\right) + O(\log \log y)\right)\right\} < \infty.$$

As a result, the probability of $\|R_y\|_\infty > \epsilon$ occuring infinitely often is zero. Therefore, almost surely the rough part of the sample of F is less than ϵ. Consequently, R_y vanishes almost surely as $y \to \infty$.

As a result, the sequence of continuous functions (S_y) uniformly converges to its limit, which by the Uniform Limit Theorem must be continuous. By uniform convergence we can compute the Fourier expansion, which recovers exactly what we expect. We defined F as the limit of its smooth parts, so therefore any samples of F are almost surely continuous. $\square$

At the start of this section, we defined $F(t)$ as the limit as $y \to \infty$ of the smooth sum

$$\frac{\eta}{\pi} \sum_{\substack{n \neq 0 \\ P^+(|n|) \leq y}} \frac{1 - e(nt)}{n} X_n.$$

Since the Fourier coefficients are bounded by $O(1/n)$ and F is almost surely a Fourier series of a continuous function, all finite Fourier sums converge to F uniformly [11]. Consequently, we can also define the process F as the limit as $N \to \infty$ of the partial symmetrical sums

$$\frac{\eta}{\pi} \sum_{|n| \leq N} \frac{1 - e(nt)}{n} X_n.$$

Therefore for the rest of the paper we can interchangeably use either definition for the infinite series $F(t)$.

3. CONVERGENCE OF FINITE-DIMENSIONAL DISTRIBUTIONS OF $\mathcal{F}_q$

In order to prove Theorem 1.1, we will first show convergence of finite-dimensional distributions. We take $(\mathcal{F}_{q,\pm}(t))_{q \text{ prime}}$ as the sequence of distributions of character paths modulo q dependent on the parity of the characters. Let $F_\pm(t)$ be random processes defined by

$$F_+(t) := \frac{\eta}{\pi} \sum_{k \geq 1} \frac{X_k}{k} \sin(2\pi kt) \qquad \text{and} \qquad F_-(t) := \frac{\eta}{\pi} \sum_{k \geq 1} \frac{X_k}{k} (1 - \cos(2\pi kt)),$$

where X_n are Steinhaus random multiplicative functions, defined in Definition 1.2, and η is uniformly distributed on the unit circle.

Theorem 3.1. *Let q be an odd prime. The sequence of the distributions of character paths $(\mathcal{F}_{q,\pm}(t))_q$ converges to the process $F_\pm(t)$ in the sense of convergence of finite distributions. In other words, for every $n \geq 1$ and for every n-tuple $0 \leq t_1 < \cdots < t_n \leq 1$, the vectors*

$$(\mathcal{F}_{q,\pm}(t_1), \ldots, \mathcal{F}_{q,\pm}(t_n))$$

converge in law as $q \to \infty$ through the primes to

$$(F_\pm(t_1), \ldots, F_\pm(t_n)).$$

We prove this by the method of moments. We will define a moment M_q, of our distribution $\mathcal{F}_q$ and a moment M for the random process F. In Section 3.3, we prove M is determinate. Subsequently, in Section 3.4, we prove this sequence of moments M_q converges to M, the moment of F. This is sufficient to prove Theorem 3.1.

We are considering odd and even character paths separately. In this proof we will focus on results for odd character paths as the proof is analogous for the even character case. Where this is not the case, any changes will be addressed throughout the section.

3.1. **Definitions of the Moments.** The Fourier series of the character path is

$$f_\chi(t) = \frac{\tau(\chi)}{2\pi i \sqrt{q}} \sum_{0 < |k| < q} \frac{\overline{\chi}(k)}{k} (1 - e(-kt)) + O\left(\frac{\log q}{\sqrt{q}}\right).$$

This results from truncating the Fourier series of the character sum $S_\chi(t)$ and the trivial inequality $|f_\chi(t) - S_\chi(t)| \leq \frac{1}{\sqrt{q}}$. The paths of odd and even characters are shown to differ greatly, exemplified in Figure 1, due to the constant term vanishing when χ is even. As such, this paper will assess distributions of these character paths modulo odd prime q separately, dependent on parity. As a Fourier series we split this into

$$f_\chi(t) = \begin{cases} \frac{-\tau(\chi)}{\pi \sqrt{q}} \sum_{k=1}^{q} \frac{\overline{\chi}(k)}{k} \sin(2\pi kt) + O\left(\frac{\log q}{\sqrt{q}}\right), & \chi \text{ even,} \\ \frac{\tau(\chi)}{\pi i \sqrt{q}} \sum_{k=1}^{q} \frac{\overline{\chi}(k)}{k} (1 - \cos(2\pi kt)) + O\left(\frac{\log q}{\sqrt{q}}\right), & \chi \text{ odd.} \end{cases}$$

We define our moments M_q and M. In this section we will assume χ is odd as the proof is analogous to the even case. Therefore, taking a function from the odd distribution $\mathcal{F}_{q,-}$, we will take the character path modulo q as

$$f_\chi(t) = \frac{\tau(\chi)}{\pi i \sqrt{q}} \sum_{1 \le n \le q} \frac{\overline{\chi}(n)}{n}(1 - \cos(2\pi n t)) + O\left(\frac{\log q}{\sqrt{q}}\right).$$

We will also be considering the odd random series

$$F_-(t) = \frac{\eta}{\pi} \sum_{n \ge 1} \frac{X_k}{k}(1 - \cos(2\pi n t)),$$

which for ease of notation will be referred to as $F(t)$ for the rest of this section.

Definition 3.1. *Let $k \ge 1$ be given and $\underline{t} = (t_1, \ldots, t_k)$, where $0 \le t_1 < \cdots < t_k \le 1$, be fixed. Additionally fix $\underline{n} = (n_1, \ldots, n_k)$ and $\underline{m} = (m_1, \ldots, m_k)$, where $n_i, m_i \in \mathbb{Z}_{\ge 0}$. We define the moment sequence $M_q(\underline{n}, \underline{m})$ as*

$$M_q(\underline{n}, \underline{m}) = \frac{2}{\phi(q)} \sum_{\chi \ odd} \prod_{i=1}^{k} f_\chi(t_i)^{n_i} \overline{f_\chi(t_i)}^{m_i},$$

and the moment $M(\underline{n}, \underline{m})$ as

$$M(\underline{n}, \underline{m}) = \mathbb{E}\left(\prod_{i=1}^{k} F(t_i)^{n_i} \overline{F(t_i)}^{m_i}\right).$$

The moment $M(\underline{n}, \underline{m})$ is well defined. To show this we prove the equivalent result that $\mathbb{E}(|F(t)|^n)$ is bounded for all n. By Fubini's theorem,

$$\mathbb{E}\left(|F(t)|^n\right) = \int_0^\infty n x^{n-1} \mathbb{P}\left(|F(t)| > x\right) dx.$$

We then use a result by Bober, Goldmakher, Granville and Koukoulopoulos [2]: Let $c = e^{-\gamma} \log 2$. For any $\tau \ge 1$,

$$\mathbb{P}\left(\max_{0 \le t \le 1} |F(t)| > 2e^\gamma \tau\right) \le \exp\left\{-\frac{e^{\tau - c - 2}}{\tau}\left(1 + O\left(\frac{\log \tau}{\tau}\right)\right)\right\}.$$

Therefore, combining both equations, the moment is finite and well defined.

$F(t)$ is a random process, defined by the almost surely converging sum

$$F(t) = \frac{\eta}{\pi} \sum_{a \ge 1} \frac{X_a}{a}(1 - \cos(2\pi a t)).$$

As shown in Section 2 we can define F as the limit of the symmetric partial sums. The infinite series F is not absolutely convergent, so justification is needed to manipulate the product $\prod_{i=1}^{k} F(t_i)^{n_i} \overline{F(t_i)}^{m_i}$.

We write the expansion of $F(t_i)^{n_i}$ as

$$\frac{\eta^{n_i}}{\pi^{n_i}} \sum_{a_{i,1}, \ldots, a_{i,n_i} \ge 1} \prod_{j=1}^{n_i} \frac{X_{a_{i,j}}}{a_{i,j}}(1 - \cos(2\pi a_{i,j} t_i)),$$

and $\overline{F(t_i)}^{m_i}$ in a similar manner. Without changing the order of summation, the product $\prod_{i=1}^{k} F(t_i)^{n_i} \overline{F(t_i)}^{m_i}$ is therefore

$$\frac{\eta^n \overline{\eta}^m}{\pi^{n+m}} \sum \cdots \sum \prod_{i=1}^{k} \prod_{j=1}^{n_i} \prod_{j'=1}^{m_i} \frac{X_{a_{i,j}} \overline{X_{b_{i,j'}}}}{a_{i,j} b_{i,j'}}(1 - \cos(2\pi a_{i,j} t_i))(1 - \cos(2\pi b_{i,j'} t_i)),$$

where $n = |\underline{n}|$ and $m = |\underline{m}|$ as above. The sums are over $a_{i,j} \ge 1$ and $b_{i,j'} \ge 1$, where $j \in [1, n_i]$ and $j' \in [1, m_i]$ for $i \in [1, k]$.

The moment $M(\underline{n}, \underline{m})$ is the expectation of this multivariate sum. To simplify the equation, we want to swap the order of expectation with the order of summation. Since the

moment is finite, we use Lebesgue's dominated convergence theorem [9, Chapter 2, Corollary 5.3] to bring the expectation inside the sum. Using the multiplicativity of Steinhaus random multiplicative functions, the moment M therefore equals[2]

$$\mathbb{E}\left(\frac{\eta^n \overline{\eta}^m}{\pi^{n+m}}\right) \sum \cdots \sum \mathbb{E}\left(X_a \overline{X_b}\right) \prod_{i=1}^{k} \prod_{j=1}^{n_i} \prod_{j'=1}^{m_i} \frac{(1-\cos(2\pi a_{i,j} t_i))(1-\cos(2\pi b_{i,j'} t_i))}{a_{i,j} b_{i,j'}},$$

where

$$a := \prod_{i=1}^{k} \prod_{j=1}^{n_i} a_{i,j}, \quad b := \prod_{i=1}^{k} \prod_{j'=1}^{m_i} b_{i,j'}.$$

Steinhaus random multiplicative functions X_n are orthogonal as n can always be written as a unique prime factorisation and $\mathbb{E}(X_p) = 0$ for all primes p. In other words,

$$\mathbb{E}\left(X_a \overline{X_b}\right) = \mathbb{1}_{a=b} := \begin{cases} 1, & a = b \\ 0, & \text{otherwise} \end{cases}.$$

Therefore we can rewrite the moment as follows,

$$M = \mathbb{E}\left(\frac{\eta^n \overline{\eta}^m}{\pi^{n+m}}\right) \sum_{l=1}^{\infty} \sum_{a=b=l} \frac{1}{ab} \prod_{i=1}^{k} \prod_{j=1}^{n_i} \prod_{j'=1}^{m_i} (1-\cos(2\pi a_{i,j} t_i))(1-\cos(2\pi b_{i,j'} t_i)),$$

where a and b are the product of $a_{i,j}$ and $b_{i,j'}$ respectively. Taking $\frac{1}{ab} = \frac{1}{l^2}$ and bounding $(1 - \cos(x)) \leq 2$, M is clearly bounded as a function of n_i and m_i. These variables are fixed and finite, so the moment is absolutely convergent. Therefore, we can swap the order of summation. As a result,

$$(4) \qquad M(\underline{n}, \underline{m}) = \mathbb{E}\left(\frac{\eta^n \overline{\eta}^m}{\pi^{n+m}}\right) \sum_{a \geq 1} \mathcal{B}_{\underline{n}, \underline{t}}(a) \mathcal{B}_{\underline{m}, \underline{t}}(a),$$

where

$$(5) \qquad \mathcal{B}_{\underline{N}, \underline{t}}(x) := \sum_{x_1 \cdots x_k = x} \prod_{i=1}^{k} \beta_{N_i, t_i}(x_i)$$

and

$$(6) \qquad \beta_{N_i, t_i}(x_i) = \sum_{y_1 \cdots y_{N_i} = x_i} \frac{1}{x_i} \prod_{j=1}^{N_i} (1-\cos(2\pi y_j t_i)).$$

The moment $M_q(\underline{n}, \underline{m})$ can be also be rewritten using methods from Bober and Goldmakher [3]. First, we use the Fourier expansion of $f_\chi(t)$, so

$$f_\chi(t_i)^{n_i} \overline{f_\chi(t_i)}^{m_i} = \frac{\tau(\chi)^{n_i} \overline{\tau(\chi)}^{m_i}}{(\pi\sqrt{q})^{n_i+m_i} i^{n_i-m_i}} \sum_{\substack{1 \leq a \leq q^{n_i} \\ 1 \leq b \leq q^{m_i}}} \overline{\chi}(a)\chi(b)\beta_{n_i,q,t_i}(a)\beta_{m_i,q,t_i}(b) + O\left(\frac{(\log q + 1)^{n_i+m_i}}{\sqrt{q}}\right),$$

where $\beta_{N,q,t}$ is defined as

$$(7) \qquad \beta_{N,q,t}(x) := \sum_{\substack{x_1 \cdots x_N = x \\ x_i \leq q}} \frac{1}{x} \prod_{k=1}^{N} (1-\cos(2\pi x_k t)),$$

for $(x, q) = 1$ and 0 otherwise[3].

Continuing to expand $M_q(\underline{n}, \underline{m})$, we take a product of all $f_\chi(t_i)^{n_i} \overline{f_\chi(t_i)}^{m_i}$ for $i \in [1, k]$, showing

$$\prod_{i=1}^{k} f_\chi(t_i)^{n_i} \overline{f_\chi(t_i)}^{m_i} = \frac{\tau(\chi)^n \overline{\tau(\chi)}^m}{(\pi\sqrt{q})^{n+m} i^{n-m}} \sum_{\substack{1 \leq a \leq q^n \\ 1 \leq b \leq q^m \\ \text{for } i \in [1,k]}} \overline{\chi}(a)\chi(b)\mathcal{B}_{\underline{n},q,\underline{t}}(a)\mathcal{B}_{\underline{m},q,\underline{t}}(b) + O\left(\frac{(\log q)^{n+m}}{\sqrt{q}}\right),$$

---

[2]Note the moment will be different for F_+, where we have $\sin(2\pi at)$ instead of $(1 - \cos(2\pi at))$.

[3]For even characters, $\beta_{N,q,t}$ would instead sum over the product of $\sin(2\pi x_k t)$.

where
$$n := n_1 + \cdots + n_k \text{ and } m := m_1 + \cdots + m_k$$
and

(8)
$$\mathcal{B}_{\underline{N},q,\underline{t}}(x) := \sum_{\substack{x_1 \cdots x_k = x \\ x_i \leq q^{N_i}}} \prod_{i=1}^{k} \beta_{N_i,q,t_i}(x_i),$$

for $(x,q) = 1$ and 0 otherwise. Note that $\mathcal{B}_{\underline{N},\underline{t}}$ and β_{N_i,t_i} from Equations (5) and (6) are the limits as $q \to \infty$ of $\mathcal{B}_{\underline{N},q,\underline{t}}$ respectively. Furthermore, we take the average of this product over all odd Dirichlet characters χ to find

(9)
$$M_q(\underline{n}, \underline{m}) = \frac{1}{(\pi\sqrt{q})^{n+m} i^{n-m}} \sum_{\substack{1 \leq a \leq q^n \\ 1 \leq b \leq q^m}} \left(\mathcal{B}_{\underline{n},q,\underline{t}}(a) \mathcal{B}_{\underline{m},q,\underline{t}}(b) \right) \frac{2}{\phi(q)} \sum_{\substack{\chi \mod q \\ \chi \text{ odd}}} \overline{\chi}(a)\chi(b)\tau(\chi)^n \overline{\tau(\chi)}^m$$
$$+ O\left(\frac{(\log q)^{n+m}}{\sqrt{q}} \right).$$

This form is more useful for future calculations and will used to prove M_q tends to M as $q \to \infty$ through the primes.

3.2. **Bounding the Moments.** Later in the paper we will be interested in bounding $\mathcal{B}_{\underline{N},q,\underline{t}}$ and $\mathcal{B}_{\underline{N},\underline{t}}$. The inequality we find is independent of q, so we can consider both bounds at the same time. Therefore for this subsection we will work with $\mathcal{B}_{\underline{N},q,\underline{t}}$.

Recall,
$$\mathcal{B}_{\underline{N},q,\underline{t}}(x) = \sum_{\substack{x_1 \cdots x_k = x \\ x_i \leq q^{N_i}}} \prod_{i=1}^{k} \beta_{N_i,q,t_i}(x_i),$$

where
$$\beta_{N,q,t}(x_i) = \sum_{\substack{y_1 \cdots y_N = x_i \\ y_j \leq q}} \frac{1}{x_i} \prod_{j=1}^{N} (1 - \cos(2\pi y_j t)),$$

for $(x_i, q) = 1$ and 0 otherwise. Since $|1 - \cos(2\pi y_j t)| \leq 2$, we always have the bound
$$|\beta_{N,q,t}(x)| \leq \frac{2^N d_N(x)}{x},$$

where $d_N(x)$ is the Nth divisor function $\sum_{x_1 \cdots x_N = x} 1$[4]. As a result,
$$\mathcal{B}_{\underline{N},q,\underline{t}}(x) \leq \frac{2^N}{x} \sum_{\substack{x_1 \cdots x_k = x \\ x_i \leq q^{N_i}}} \prod_{i=1}^{k} d_{N_i}(x_i),$$

where $N = \sum N_i = |\underline{N}|$. To further bound $\mathcal{B}$ we next use the following lemma.

Lemma 3.2. *Let $d_{N_1}(x_1), d_{N_2}(x_2)$ be the N_1th and N_2th divisor function of $x_1, x_2 \in \mathbb{N}$ respectively. We have the relation*
$$d_{N_1}(x_1) d_{N_2}(x_2) \leq d_{N_1+N_2}(x_1 \cdot x_2).$$

Proof. We apply a combinatorial argument, where we view $d_N(x)$ as the number of ways of choosing N positive integers that multiply to x. Therefore $d_{N_1+N_2}(x_1 \cdot x_2)$ is at least the number of ways of choosing N_1 integers multiplying to x_1 times the number of ways of choosing N_2 integers multiplying to x_2. □

---

[4]For F_+ and even character paths, the 2^N vanishes in the bound of β as $|\sin(2\pi y_j t)| \leq 1$. However, since this bound is only included in error terms, the difference of the constant is irrelevant.

Using Lemma 3.2, we bound $\mathcal{B}_{\underline{N},q,\underline{t}}(x)$ by

$$(10) \qquad \mathcal{B}_{\underline{N},q,\underline{t}}(x) \leq \frac{2^N d_N(x)}{x} \sum_{\substack{x_1 \cdots x_k = x \\ x_i \leq q^{N_i}}} 1 \leq \frac{2^N d_N(x) d_k(x)}{x}$$

$$\leq \frac{2^N d^{N+k}(x)}{x}.$$

In parts of the proof, it is sufficient to use the looser bound, however we will mainly apply the bound from Equation (10). This will be useful in future equations. Note that this is independent of q and $\underline{t}$, so the bounds hold for $\mathcal{B}_{\underline{N},\underline{t}} = \lim_{q \to \infty} \mathcal{B}_{\underline{N},q,\underline{t}}$.

3.3. **Proving Determinacy.** Our aim is to use the method of moments to prove the distribution of character paths modulo q converges to $F(t)$ in the sense of finite distributions. For this we need to show the moment $M(\underline{n}, \underline{m})$ is determinate, or in other words show the moment only has one representing measure. To show that M is a determinate complex moment sequence, it is sufficient to show that it satisfies

$$(11) \qquad \sum_{n=1}^{\infty} M(\underline{n}, \underline{n})^{-1/2n} = \infty,$$

where $\underline{n} = (n_1, \ldots, n_k)$ and $n = |\underline{n}| = \sum_i n_i$. This is also known as the Carleman condition [16, Theorem 15.11].

Lemma 3.3. *The moment $M(\underline{n}, \underline{m})$ satisfies Equation (11).*

Proof. This is shown using Equation (4) and taking $\underline{n} = \underline{m}$. Setting $n = |\underline{n}| = |\underline{m}|$, we have

$$M(\underline{n}, \underline{m}) = \frac{1}{\pi^{2n}} \sum_{a \geq 1} \mathcal{B}_{\underline{n},\underline{t}}(a)^2.$$

We use the bound of $\mathcal{B}$ from Equation (10), taking $d_k(a) \leq a^{\epsilon_k}$ for small $\epsilon_k > 0$, so

$$M(\underline{n}, \underline{n}) \leq \frac{2^{2n}}{\pi^{2n}} \sum_{a \geq 1} \frac{d_n(a)^2}{a^{2-2\epsilon_k}} =: \frac{2^{2n}}{\pi^{2n}} \sum_{a \geq 1} \frac{d_n(a)^2}{a^{2\sigma}},$$

taking $\sigma := 1 - \epsilon_k$. We can use Proposition 3.2 from Bober and Goldmakher [3], which states for $1/2 < \sigma \leq 1$,

$$(12)$$
$$\sum_{a=1}^{\infty} \frac{d_n(a)^2}{a^{2\sigma}} \leq \exp\left(2n\sigma \log\log(2n)^{1/\sigma} + \frac{(2n)^{1/\sigma}}{2\sigma - 1} + O\left(\frac{n}{2\sigma - 1} + \frac{(2n)^{1/\sigma}}{\log(3(2n)^{1/\sigma-1})}\right)\right).$$

Here we have shown the sum has the lower bound

$$\sum_{n=1}^{\infty} M(\underline{n}, \underline{n})^{-1/2n} \geq \frac{\pi}{2} \sum_{n=1}^{\infty} \exp\left(-\sigma \log\log\left((2n)^{1/\sigma}\right) - \frac{(2n)^{1/\sigma-1}}{2\sigma - 1} + O\left(\frac{1}{2\sigma - 1} + \frac{(2n)^{1/\sigma-1}}{\log(3(2n)^{1/\sigma-1})}\right)\right).$$

The lower bound can be rewritten as

$$\frac{\pi}{2} \sum_{n=1}^{\infty} \frac{\sigma^\sigma}{(\log 2n)^\sigma} \exp\left(-\frac{(2n)^{\frac{1-\sigma}{\sigma}}}{2\sigma - 1}\right) \exp\left(O\left(\frac{1}{2\sigma - 1} + \frac{(2n)^{1/\sigma-1}}{\log(3(2n)^{1/\sigma-1})}\right)\right).$$

Tending $\sigma = 1 - \epsilon_k$ to 1, this sum diverges. Therefore

$$\sum_{n=1}^{\infty} M(\underline{n}, \underline{n})^{-1/2n} = \infty,$$

and the Carleman condition holds. Therefore the claim is proved. $\square$

3.4. **Convergence of Moments.** In this section we show the moment sequence M_q converges to the multivariate moment of F, therefore proving Theorem 3.1. Separating the distribution by parity, we have two propositions.

Proposition 3.4. *Let $k \geq 1$ be given and $0 \leq t_1 < \cdots < t_k \leq 1$ be fixed. Fix $\underline{n} = (n_1, \ldots, n_k)$ and $\underline{m} = (m_1, \ldots, m_k)$, where $n_i, m_i \in \mathbb{Z}_{\geq 0}$. Let*

$$M_{q,-}(\underline{n}, \underline{m}) = \frac{2}{\phi(q)} \sum_{\chi \ odd} \prod_{i=1}^{k} f_\chi(t_i)^{n_i} \overline{f_\chi(t_i)}^{m_i}.$$

Then for all $\epsilon > 0$,

$$M_{q,-}(\underline{n}, \underline{m}) = M_-(\underline{n}, \underline{m}) + O_{\underline{n}, \underline{m}, k} \left(q^{-1/2 + \epsilon} \right),$$

where

$$M_-(\underline{n}, \underline{m}) = \mathbb{E} \left(\prod_{i=1}^{k} F_-(t_i)^{n_i} \overline{F_-(t_i)}^{m_i} \right).$$

Importantly, $M_{q,-}(\underline{n}, \underline{m}) \to M_-(\underline{n}, \underline{m})$ as $q \to \infty$ through the primes.

Proposition 3.5. *Let $k \geq 1$ be given and fix $\underline{t}, \underline{n}, \underline{m}$ as in Proposition 3.4. Let*

$$M_{q,+}(\underline{n}, \underline{m}) = \frac{2}{\phi(q)} \sum_{\chi \ even} \prod_{i=1}^{k} f_\chi(t_i)^{n_i} \overline{f_\chi(t_i)}^{m_i}.$$

Then for all $\epsilon > 0$,

$$M_{q,+}(\underline{n}, \underline{m}) = M_-(\underline{n}, \underline{m}) + O_{\underline{n}, \underline{m}, k} \left(q^{-1/2 + \epsilon} \right),$$

where

$$M_+(\underline{n}, \underline{m}) = \mathbb{E} \left(\prod_{i=1}^{k} F_+(t_i)^{n_i} \overline{F_+(t_i)}^{m_i} \right).$$

Importantly, $M_{q,+}(\underline{n}, \underline{m}) \to M_+(\underline{n}, \underline{m})$ as $q \to \infty$ through the primes.

In this section we only look at the $\mathcal{F}_{q,-}$ case, where χ is odd. There are equivalent propositions and lemmas for the even case, where the proofs are analogous to the proofs shown in the section. In places where the proof differs, we will state the results for $\mathcal{F}_{q,+}$ and how it doesn't largely affect the proof.

These propositions are sufficient to prove Theorem 3.1, showing $(\mathcal{F}_q(t))_{q \text{ prime}}$ converges in finite distributions to $F(t)$. We prove Proposition 3.4 as a combination of the following 2 propositions.

Proposition 3.6. *Let $k \geq 1$ be given and $\underline{t} = (t_1, \ldots, t_k)$, where $0 \leq t_1 < \cdots < t_k \leq 1$, be fixed. Fix $\underline{n} = (n_1, \ldots, n_k)$ and $\underline{m} = (m_1, \ldots, m_k)$, where $n_i, m_i \in \mathbb{Z}_{\geq 0}$ and*

$$n := n_1 + n_2 + \cdots + n_k = m_1 + \cdots + m_k.$$

The moment sequence defined in Proposition 3.4 can be expressed as

$$M_{q,-}(\underline{n}, \underline{m}) = \frac{1}{\pi^{2n}} \sum_{a \geq 1} \mathcal{B}_{\underline{n}, \underline{t}}(a) \mathcal{B}_{\underline{m}, \underline{t}}(a) + O_{\underline{n}, \underline{m}, k} \left(\frac{(\log q)^{2n}}{\sqrt{q}} \right),$$

where $\mathcal{B}_{\underline{N}, \underline{t}}$ is defined as

$$\mathcal{B}_{\underline{N}, \underline{t}}(a) = \frac{1}{a} \sum_{x_1 \cdots x_k = a} \prod_{i=1}^{k} \left(\sum_{y_1 \cdots y_N = x_i} \prod_{j=1}^{N_i} (1 - \cos(2\pi y_j t)) \right).$$

Proposition 3.7. *Let $k \geq 1$ be given and $\underline{t} = (t_1, \ldots, t_k)$, where $0 \leq t_1 < \cdots < t_k \leq 1$, be fixed. Fix $\underline{n} = (n_1, \ldots, n_k)$ and $\underline{m} = (m_1, \ldots, m_k)$, where $n_i, m_i \in \mathbb{Z}_{\geq 0}$ and*

$$n := n_1 + n_2 + \cdots + n_k = m_1 + \cdots + m_k.$$

Then

$$M_-(\underline{n}, \underline{m}) = \frac{1}{\pi^{2n}} \sum_{a \geq 1} \mathcal{B}_{\underline{n}, \underline{t}}(a) \mathcal{B}_{\underline{m}, \underline{t}}(a),$$

where $\mathcal{B}_{\underline{N}, \underline{t}}$ is defined as in Proposition 3.6.

Before the proof of the propositions, we will use them to prove Proposition 3.4.

Proof of Proposition 3.4. Take $n = n_1 + n_2 + \cdots + n_k$ and $m = m_1 + \cdots m_k$. We split the proof into 2 cases: $n = m$ and $n \neq m$. The first case has already been shown by Propositions 3.6 and 3.7:

$$M_{q,-}(\underline{n}, \underline{m}) = \frac{1}{\pi^{2n}} \sum_{a \geq 1} \mathcal{B}_{\underline{n}, \underline{t}}(a) \mathcal{B}_{\underline{m}, \underline{t}}(a) + O_{\underline{n}, \underline{m}, k}\left(\frac{(\log q)^{2n}}{\sqrt{q}}\right)$$

$$= \mathbb{E}\left(\prod_{i=1}^{k} F(t_i)^{n_i} \overline{F(t_i)}^{m_i}\right) + O_{\underline{n}, \underline{m}, k}\left(\frac{(\log q)^{2n}}{\sqrt{q}}\right).$$

Therefore, the only case left to show is when $n \neq m$. We recall Equation (4):

$$M_-(\underline{n}, \underline{m}) = \mathbb{E}\left(\frac{\eta^n \overline{\eta}^m}{\pi^{n+m}}\right) \sum_{a \geq 1} \mathcal{B}_{\underline{n}, \underline{t}}(a) \mathcal{B}_{\underline{m}, \underline{t}}(a).$$

Since η is uniformly distributed on the unit circle, $\mathbb{E}(\eta^n \overline{\eta}^m) = 0$ and the moment M_- vanishes. Therefore, to conclude the proof, we need to show the moment $M_{q,-} \to 0$ as $q \to \infty$. As shown in Equation (9), we can write $M_{q,-}(\underline{n}, \underline{m})$ as

$$M_{q,-}(\underline{n}, \underline{m}) = \frac{1}{(\pi\sqrt{q})^{n+m}} \sum_{\substack{1 \leq a \leq q^n \\ 1 \leq b \leq q^m}} \left(\mathcal{B}_{\underline{n}, q, \underline{t}}(a) \mathcal{B}_{\underline{m}, q, \underline{t}}(b)\right) \frac{2}{\phi(q)} \sum_{\chi} \overline{\chi}(a) \chi(b) \tau(\chi)^n \overline{\tau(\chi)}^m$$

$$+ O\left(\frac{(\log q)^{n+m}}{\sqrt{q}}\right).$$

Assuming $n > m$, we rewrite $\tau(\chi)^n \overline{\tau(\chi)}^m$ as $q^m \tau(\chi)^{n-m}$. Therefore, taking $\chi(\bar{a}) := \overline{\chi}(a)$,

$$\frac{2}{\phi(q)} \sum_{\chi} \overline{\chi}(a) \chi(b) \tau(\chi)^n \overline{\tau(\chi)}^m = \frac{2q^m}{\phi(q)} \sum_{\chi} \chi(\bar{a} \cdot b) \tau(\chi)^{n-m}.$$

Lemma 3.8. *For $N \in \mathbb{N}$,*

$$\frac{2}{\phi(q)} \left| \sum_{\substack{\chi \bmod q \\ \chi(-1) = \sigma}} \chi(a) \tau(\chi)^N \right| \leq 2N q^{(N-1)/2},$$

where $\sigma = \{1, -1\}$.

This lemma is a slight generalisation of a result by Granville and Soundararajan [8, Lemma 8.3] and uses Deligne's bound on hyper-Kloosterman sums. Below follows Granville and Soundararajan's proof, with a modification to include when χ is even.

Proof. Firstly, we rewrite the sum as exponential sums, using orthogonality of characters and the definition of the Gauss sum $\tau(\chi)$:

$$\frac{2}{\phi(q)} \sum_{\substack{\chi \bmod q \\ \chi(-1) = \sigma}} \chi(a) \tau(\chi)^N = \sum_{\substack{x_1, \ldots, x_N \bmod q \\ x_1 \cdots x_N \equiv a \bmod q}} e\left(\frac{x_1 + \cdots + x_N}{q}\right) + \mathrm{sgn}(\sigma) \sum_{\substack{x_1, \ldots, x_N \bmod q \\ x_1 \cdots x_N \equiv -a \bmod q}} e\left(\frac{x_1 + \cdots + x_N}{q}\right).$$

Then, using Deligne's bound [5]

$$\left| \sum_{\substack{x_1,\ldots,x_N \mod q \\ x_1 \cdots x_N \equiv b \mod q}} e\left(\frac{x_1 + \cdots + x_N}{q}\right) \right| \leq N q^{(N-1)/2},$$

we have proved the lemma. $\qquad \square$

As a result, we have the inequality

$$|M_{q,-}(\underline{n},\underline{m})| \leq \frac{2(n-m)}{\pi^{(n+m)}\sqrt{q}} \sum_{\substack{1 \leq a \leq q^n \\ 1 \leq b \leq q^m}} |\mathcal{B}_{\underline{n},q,\underline{t}}(a)||\mathcal{B}_{\underline{m},q,\underline{t}}(b)| + O\left(\frac{(\log q)^{n+m}}{\sqrt{q}}\right).$$

We also have the bound on $\mathcal{B}$, as shown in Equation (10),

$$\mathcal{B}_{\underline{N},q,\underline{t}}(x) \leq \frac{2^N d_N(x) d_k(x)}{x}.$$

Therefore, trivially bounding both divisor functions by q^ϵ for $\epsilon > 0$,

$$\sum_{1 \leq a \leq q^n} |\mathcal{B}_{\underline{n},q,\underline{t}}(a)| \ll 2^n q^\epsilon \sum_{1 \leq a \leq q^n} \frac{1}{a} \leq 2^n q^\epsilon \log(q^n).$$

We get an analogous result for $\sum_{1 \leq b \leq q^m} |\mathcal{B}_{\underline{m},q,\underline{t}}(b)|$. As a result,

$$M_{q,-}(\underline{n},\underline{m}) \ll \frac{2^{1+n+m}(n-m)q^\epsilon \log(q^n)\log(q^m)}{\pi^{n+m}\sqrt{q}} + O\left(\frac{(\log q)^{n+m}}{\sqrt{q}}\right) \ll_{n,m} q^{-1/2+\epsilon},$$

which tends to zero as $q \to \infty$. By a similar method we can show this is also the case when $n < m$. Therefore Proposition 3.4 holds. $\qquad \square$

Having proven Proposition 3.4 assuming Propositions 3.6 and 3.7, we will now prove both results, showing when $|\underline{n}| = |\underline{m}|$ both $\lim_{q\to\infty} M_q$ and M equal

$$\frac{1}{\pi^{2n}} \sum_{a \geq 1} \mathcal{B}_{\underline{n},\underline{t}}(a) \mathcal{B}_{\underline{m},\underline{t}}(a).$$

Recall that we are only proving the odd character case, as the proof is analogous for $\mathcal{F}_{q,+}$. For odd characters, $\mathcal{B}_{N,q,t}(x)$ is defined in Section 3.1 as

$$\sum_{\substack{x_1 \cdots x_k = x \\ x_i \leq q^{N_i}}} \prod_{i=1}^{k} \beta_{N_i,q,t_i}(x_i) = \sum_{\substack{x_1 \cdots x_k = x \\ x_i \leq q^{N_i}}} \frac{1}{x} \prod_{i=1}^{k} \sum_{\substack{y_1 \cdots y_{N_i} = x_i \\ y_i \leq q}} \prod_{j=1}^{N_i} (1 - \cos(2\pi y_j t)).$$

For ease of notation, in the proofs we refer to $M_{q,-}$ and M_- as M_q and M respectively.

Proof of Proposition 3.6. Taking $n = m$, where

$$n := |\underline{n}| = n_1 + \cdots + n_k, \qquad\qquad m := |\underline{m}| = m_1 + \cdots + m_k,$$

we rewrite Equation (9) to

$$M_q(\underline{n},\underline{m}) = \frac{1}{\pi^{2n}} \sum_{1 \leq a,b \leq q^n} \left(\mathcal{B}_{\underline{n},q,\underline{t}}(a)\mathcal{B}_{\underline{m},q,\underline{t}}(b)\right) \frac{2}{\phi(q)} \sum_{\chi \text{ odd}} \overline{\chi}(a)\chi(b) + O\left(\frac{(\log q)^{2n}}{\sqrt{q}}\right),$$

where $\mathcal{B}_{\underline{n},q,\underline{t}}$ is defined as in Equation (8).

Using the orthogonality of χ, and noting we are only summing over odd characters χ modulo q[5], the moment sequence becomes

(13) $$M_q(\underline{n},\underline{m}) = \frac{1}{\pi^{2n}}\Sigma_+(\underline{n},\underline{m}) - \frac{1}{\pi^{2n}}\Sigma_-(\underline{n},\underline{m}) + O\left(\frac{(\log q)^{2n}}{\sqrt{q}}\right),$$

---

[5]The method for the even character case would differ here. Firstly recall $\mathcal{B}$ involves $\sin$ instead of $(1-\cos)$ in the even case. Additionally, since these are even characters, we would have $M_q = \frac{1}{\pi^{2n}}\Sigma_+ + \frac{1}{\pi^{2n}}\Sigma_- + O\left((\log q)^{2n}q^{-1/2}\right)$, where we are adding the Σ_- term instead of subtracting it. However, the Σ_- term is eventually swallowed by the error term, so this doesn't affect the end result.

where

$$\Sigma_+(\underline{n},\underline{m}) := \sum_{\substack{1 \le a,b \le q^n \\ a \equiv b \mod q}} \mathcal{B}_{\underline{n},q,\underline{t}}(a)\mathcal{B}_{\underline{m},q,\underline{t}}(b), \quad \Sigma_-(\underline{n},\underline{m}) := \sum_{\substack{1 \le a,b \le q^n \\ a \equiv -b \mod q}} \mathcal{B}_{\underline{n},q,\underline{t}}(a)\mathcal{B}_{\underline{m},q,\underline{t}}(b).$$

The aim is to get the main sum independent of q. Using ideas from Bober and Gold-makher [3, Proof of Lemma 4.1], we consider Σ_+ and Σ_- simultaneously. First we split the sums into arithmetic progressions mod q, noting that $\mathcal{B}_{\underline{n},q,\underline{t}}(x)$ vanishes when $q|x$,

$$\Sigma_\pm(\underline{n},\underline{m}) = \sum_{\substack{1 \le a,b < q \\ a \equiv \pm b \mod q}} \sum_{0 \le \gamma_1,\gamma_2 < q^{n-1}} \mathcal{B}_{\underline{n},q,\underline{t}}(a + \gamma_1 q)\mathcal{B}_{\underline{m},q,\underline{t}}(b + \gamma_2 q).$$

We simplify $\Sigma_\pm$ by splitting the inner sum into $\gamma_1 = \gamma_2 = 0$, $\gamma_1 \ne 0$, and $\gamma_2 \ne 0$:

$$\sum_{0 \le \gamma_1,\gamma_2 < q^{n-1}} \mathcal{B}_{\underline{n},q,\underline{t}}(a + \gamma_1 q)\mathcal{B}_{\underline{m},q,\underline{t}}(b + \gamma_2 q)$$

$$= \left(\mathcal{B}_{\underline{n},q,\underline{t}}(a)\mathcal{B}_{\underline{m},q,\underline{t}}(b)\right) + \sum_{j=1}^{2} \sum_{\substack{0 \le \gamma_1,\gamma_2 < q^{n-1} \\ \gamma_j \ne 0}} \mathcal{B}_{\underline{n},q,\underline{t}}(a + \gamma_1 q)\mathcal{B}_{\underline{m},q,\underline{t}}(b + \gamma_2 q).$$

For ease of notation, we define the above latter sum as

$$\Omega = \sum_{j=1}^{2} \sum_{\substack{0 \le \gamma_1,\gamma_2 < q^{n-1} \\ \gamma_j \ne 0}} \mathcal{B}_{\underline{n},q,\underline{t}}(a + \gamma_1 q)\mathcal{B}_{\underline{m},q,\underline{t}}(b + \gamma_2 q).$$

We can bound Ω by using the bound of $\mathcal{B}$ shown in Section 3.2, so

$$\Omega \le 2^{2n} \sum_{j=1}^{2} \sum_{\substack{0 \le \gamma_1,\gamma_2 < q^{n-1} \\ \gamma_j \ne 0}} \frac{d_n(a + \gamma_1 q)d_k(a + \gamma_1 q)}{a + \gamma_1 q} \frac{d_n(b + \gamma_2 q)d_k(b + \gamma_2 q)}{b + \gamma_2 q}.$$

By bounding the divisor functions by $O_{n,k}(q^\epsilon)$, we can further bound the sum to

$$\Omega \ll_{n,k} q^\epsilon \sum_{j=1}^{2} \sum_{\substack{0 \le \gamma_1,\gamma_2 < q^{n-1} \\ \gamma_j \ne 0}} \frac{1}{a + \gamma_1 q} \frac{1}{b + \gamma_2 q}.$$

We can use the bound on partial harmonic series,

$$\omega_x := \sum_{\gamma=1}^{q^{n-1}} \frac{1}{x + \gamma q} \le \frac{\log(q^{n-1})}{q},$$

to further bound Ω. As a result,

$$\sum_{j=1}^{2} \sum_{\substack{0 \le \gamma_1,\gamma_2 < q^{n-1} \\ \gamma_j \ne 0}} \frac{1}{a + \gamma_1 q} \frac{1}{b + \gamma_2 q} = \left(\frac{1}{a} + \omega_a\right)\omega_b + \omega_a\left(\frac{1}{b} + \omega_b\right) \le \frac{\log(q^{n-1})}{q}\left(\frac{1}{a} + \frac{1}{b} + \frac{2\log(q^{n-1})}{q}\right).$$

Therefore $\Sigma_\pm$ can be written as,

$$\Sigma_\pm = \sum_{\substack{1 \le a,b < q \\ a \equiv \pm b \mod q}} \left(\mathcal{B}_{\underline{n},q,\underline{t}}(a)\mathcal{B}_{\underline{m},q,\underline{t}}(b)\right) + O_{n,k}\left(\frac{q^\epsilon \log(q^{n-1})}{q} \sum_{\substack{1 \le a,b < q \\ a \equiv \pm b \mod q}} \left(\frac{1}{a} + \frac{1}{b} + \frac{2\log(q^{n-1})}{q}\right)\right).$$

For Σ_+ we have $a \equiv +b \mod q$ and $1 \le a,b \le q$. Therefore $a = b$ and we have

$$\Sigma_+ = \sum_{1 \le a < q} \left(\mathcal{B}_{\underline{n},q,\underline{t}}(a)\mathcal{B}_{\underline{m},q,\underline{t}}(a)\right) + O_{n,k}\left(\frac{q^\epsilon \log(q^{n-1})}{q} \sum_{1 \le a < q} \left(\frac{2}{a} + \frac{2\log(q^{n-1})}{q}\right)\right).$$

For Σ_- we have $1 \le a, b \le q$ and $a \equiv -b \mod q$. Therefore $b = q - a$ and

$$\Sigma_- = \sum_{1 \le a < q} \left(\mathcal{B}_{\underline{n},q,\underline{t}}(a) \mathcal{B}_{\underline{m},q,\underline{t}}(q-a) \right) + O_{n,k} \left(\frac{q^\epsilon \log(q^{n-1})}{q} \sum_{1 \le a < q} \left(\frac{1}{a} + \frac{1}{q-a} + \frac{2\log(q^{n-1})}{q} \right) \right).$$

We bound the partial harmonic series again by $\log q$ to simplify both errors for Σ_+ and Σ_-. Consequently both error terms above can be bounded by

$$O_{n,k} \left(\frac{\log(q^{n-1})}{q^{1-\epsilon}} \left(2\log q + 2\log(q^{n-1}) \right) \right).$$

By combining the error terms, the moment sequence from Equation (13) is[6]

$$M_q(\underline{n}, \underline{m}) = \frac{1}{\pi^{2n}} \Sigma_+(\underline{n}, \underline{m}) - \frac{1}{\pi^{2n}} \Sigma_-(\underline{n}, \underline{m}) + O\left(\frac{(\log q)^{2n}}{\sqrt{q}} \right)$$

$$= \frac{1}{\pi^{2n}} \sum_{1 \le a \le q} \left(\mathcal{B}_{\underline{n},q,\underline{t}}(a) \mathcal{B}_{\underline{m},q,\underline{t}}(a) \right) - \frac{1}{\pi^{2n}} \sum_{1 \le a \le q} \left(\mathcal{B}_{\underline{n},q,\underline{t}}(a) \mathcal{B}_{\underline{m},q,\underline{t}}(q-a) \right)$$

$$+ O_{n,k} \left(\frac{2^{2n+2}(\log q^{n-1})^2}{q^{1-\epsilon}} \right) + O\left(\frac{(\log q)^{2n}}{\sqrt{q}} \right).$$

Our aim is to only have one main term,

$$\frac{1}{\pi^{2n}} \sum_{a \ge 1} \left(\mathcal{B}_{\underline{n},\underline{t}}(a) \mathcal{B}_{\underline{m},\underline{t}}(a) \right),$$

so we want to first bound the term

$$(14) \qquad \frac{1}{\pi^{2n}} \sum_{1 \le a \le q} \left(\mathcal{B}_{\underline{n},q,\underline{t}}(a) \mathcal{B}_{\underline{m},q,\underline{t}}(q-a) \right),$$

and then extend the sum

$$\sum_{1 \le a \le q} \left(\mathcal{B}_{\underline{n},q,\underline{t}}(a) \mathcal{B}_{\underline{m},q,\underline{t}}(a) \right)$$

over all positive integers.

To bound Expression (14) we again use the bound of $\mathcal{B}$ from Section 3.2 to show

$$\sum_{1 \le a < q} \left(\mathcal{B}_{\underline{n},q,\underline{t}}(a) \mathcal{B}_{\underline{m},q,\underline{t}}(q-a) \right) \le 2^{2n} \sum_{1 \le a < q} \frac{d_n(a)d_k(a)}{a} \frac{d_n(q-a)d_k(q-a)}{q-a}$$

$$\ll q^\epsilon \sum_{1 \le a < q} \frac{1}{a(q-a)} \le q^\epsilon \frac{2\log q}{q}.$$

As a result,

$$M_q(\underline{n}, \underline{n}) = \frac{1}{\pi^{2n}} \sum_{1 \le a \le q} \left(\mathcal{B}_{\underline{n},q,\underline{t}}(a) \mathcal{B}_{\underline{m},q,\underline{t}}(a) \right) + O_{n,k} \left(\frac{\log q}{q^{1-\epsilon}} \right) + O_{n,k} \left(\frac{(\log q^{n-1})^2}{q^{1-\epsilon}} \right) + O\left(\frac{(\log q)^{2n}}{\sqrt{q}} \right).$$

This can be simplified, as $\mathcal{B}_{\underline{n},q,\underline{t}}$ is equivalent to $\mathcal{B}_{\underline{n},\underline{t}}$ in this case, and we can combine the errors. Since $k, \underline{n}, \underline{m}$ are all fixed, we omit the dependencies on the error for ease of notation. Therefore

$$(15) \qquad M_q(\underline{n}, \underline{n}) = \frac{1}{\pi^{2n}} \sum_{1 \le a \le q} \left(\mathcal{B}_{\underline{n},\underline{t}}(a) \mathcal{B}_{\underline{m},\underline{t}}(a) \right) + O\left(\frac{(\log q)^{2n}}{\sqrt{q}} \right).$$

The final step is to extend the main sum to infinity. We rewrite the sum

$$\sum_{1 \le a \le q} \left(\mathcal{B}_{\underline{n},\underline{t}}(a) \mathcal{B}_{\underline{m},\underline{t}}(a) \right) = \sum_{a \ge 1} \left(\mathcal{B}_{\underline{n},\underline{t}}(a) \mathcal{B}_{\underline{m},\underline{t}}(a) \right) - \sum_{a > q} \left(\mathcal{B}_{\underline{n},\underline{t}}(a) \mathcal{B}_{\underline{m},\underline{t}}(a) \right).$$

---

[6]Recall we are add the Σ_- term instead of subtracting it in the even character case.

By bounding $\mathcal{B}$ as before, the second sum has the upper bound

$$\sum_{a>q} \left(\mathcal{B}_{\underline{n},\underline{t}}(a)\mathcal{B}_{\underline{m},\underline{t}}(a)\right) \leq 2^{2n} \sum_{a>q} \left(\frac{d_n^2(a)d_k^2(a)}{a^2}\right).$$

We take $d_n(a)^2 d_k(a)^2 = O(a^{2\epsilon_{k,n}}) =: O_k(a^\epsilon)$, so

$$(16) \qquad \sum_{a>q} \left(\mathcal{B}_{\underline{n},\underline{t}}(a)\mathcal{B}_{\underline{m},\underline{t}}(a)\right) \ll \sum_{a>q} \frac{a^\epsilon}{a^2} \ll q^{-1+\epsilon}.$$

This bound is clearly smaller than the error term in Equation (15), so as a result,

$$M_q(\underline{n},\underline{n}) = \frac{1}{\pi^{2n}} \sum_{a\geq 1} \left(\mathcal{B}_{\underline{n},\underline{t}}(a)\mathcal{B}_{\underline{m},\underline{t}}(a)\right) + O\left(\frac{(\log q)^{2n}}{\sqrt{q}}\right).$$

$\square$

To finish proving Proposition 3.4 we prove Proposition 3.7, showing how the expectation also equals the sum

$$\frac{1}{\pi^{2n}} \sum_{a\geq 1} \left(\mathcal{B}_{\underline{n},\underline{t}}(a)\mathcal{B}_{\underline{m},\underline{t}}(a)\right).$$

Proof of Proposition 3.7. We are interested in the expectation

$$M(\underline{n},\underline{m}) = \mathbb{E}\left(\prod_{i=1}^{k} F(t_i)^{n_i} \overline{F(t_i)}^{m_i}\right).$$

Using Equation (4) from Section 3.1, and $n := n_1 + \cdots n_k = m_1 + \cdots + m_k$, the moment is equivalent to

$$M(\underline{n},\underline{m}) = \mathbb{E}\left(\frac{\eta^n \overline{\eta}^n}{\pi^{2n}}\right) \sum_{a\geq 1} \mathcal{B}_{\underline{n},\underline{t}}(a)\mathcal{B}_{\underline{m},\underline{t}}(a).$$

Therefore we have

$$\mathbb{E}\left(\prod_{i=1}^{k} F_-(t_i)^{n_i} \overline{F_-(t_i)}^{m_i}\right) = \frac{1}{\pi^{2n}} \sum_{a\geq 1} \mathcal{B}_{\underline{n},\underline{t}}(a)\mathcal{B}_{\underline{m},\underline{t}}(a),$$

proving Proposition 3.7. $\square$

Therefore we have shown the multivariate moment sequence

$$M_q(\underline{n},\underline{m}) = \frac{2}{\phi(q)} \sum_{\chi \text{ odd}} \prod_{i=1}^{k} f_\chi(t_i)^{n_i} \overline{f_\chi(t_i)}^{m_i}$$

converges, as $q \to \infty$ through the primes, to

$$\mathbb{E}\left(\prod_{i=1}^{k} F(t_i)^{n_i} \overline{F(t_i)}^{m_i}\right),$$

for all k-tuples $\underline{n},\underline{m}$ and $0 \leq t_1 < \cdots < t_k \leq 1$. This section only addressed the odd character case, but the proof is similar for even characters and leads to the same results. Therefore $(\mathcal{F}_{q,\pm})_{q \text{ prime}}$, the distribution of odd/even character paths f_χ modulo q, converges to $F_\pm$ as $q \to \infty$ in the sense of convergence in finite distributions.

4. Relative Compactness of the Sequence of Distributions

In the previous section we showed $(\mathcal{F}_q)$ converges in finite distributions to the process F as $q \to \infty$ through the primes. If we can prove the sequence of distributions is relatively compact, then it follows that $(\mathcal{F}_q)$ converges in distribution to F [1, Example 5.1]. This is much stronger than convergence of finite-dimensional distributions and concludes the proof of Theorem 1.1.

Prohorov's Theorem [1, Theorem 5.1] states that if a sequence of probability measures is tight, then it must be relatively compact. For this we use Kolmorogorov's tightness criterion, quoted from Revuz and Yor:

Proposition 4.1. [15, Th. XIII.1.8] *Let* $(L_p(t))_{t \in [0,1]}$ *be a sequence of* $C([0,1])-valued$ *processes such that* $L_p(0) = 0$ *for all* p. *If there exists constants* $\alpha > 0$, $\delta > 0$ *and* $C \geq 0$ *such that for any* p *and any* $s < t$ *in* $[0,1]$, *and we have*

$$\mathbb{E}(|L_p(t) - L_p(s)|^\alpha) \leq C|t - s|^{1+\delta},$$

then the sequence $(L_p(t))$ *is tight.*

For our sequence of processes $(\mathcal{F}_q(t))_{t \in [0,1]}$ we have $f_\chi(0) = 0$ for all q. We also have the trivial bound

$$|f_\chi(t) - f_\chi(s)| \leq \sqrt{q}|t - s|,$$

leading to

$$\mathbb{E}|f_\chi(t) - f_\chi(s)|^{2k} \leq q^k|t - s|^{2k}.$$

As a result, for $k > 1$ if we take $|t - s| < \frac{1}{q^{1-\epsilon}}$ for $\epsilon \in (0, \frac{k-1}{2k-1})$, we have

$$(17) \qquad \mathbb{E}|f_\chi(t) - f_\chi(s)|^{2k} \leq |t - s|^{2k - \frac{k}{1-\epsilon}} =: |t - s|^{1+\delta_1},$$

where $\delta_1 := \frac{k-1+\epsilon(1-2k)}{1-\epsilon}$. Therefore if we show a similar bound for $\mathbb{E}|f_\chi(t) - f_\chi(s)|^{2k}$ for $|t - s| > \frac{1}{q^{1-\epsilon}}$ then the tightness condition holds for our sequence of processes.

Various authors have found results bounding the average of the difference of character sums. For example, Cochrane and Zheng [4] prove for positive integers k and Dirichlet characters modulo prime q,

$$\frac{1}{q-1} \sum_{\chi \neq \chi_0} \left| \sum_{n=s+1}^{s+t} \chi(n) \right|^{2k} \ll_{\epsilon,k} q^{k-1+\epsilon} + |t - s|^k q^\epsilon.$$

To prove tightness however we need the $|t - s|$ term independent of q.

Lemma 4.2. *Let* q *be an odd prime. For all* $\epsilon \in (0,1)$, *there exists absolute constants* $C_1(\epsilon), C_2$ *independent of* q *such that for all* $0 \leq s < t \leq 1$,

$$\mathbb{E}|f_\chi(t) - f_\chi(s)|^4 \leq C_1(\epsilon)|t - s|^{1+\delta_2} + C_2 \frac{(\log q)^4}{q},$$

where $\delta_2 := 1 - \epsilon$.

This lemma can be applied to characters of all moduli, not just primes, but for our work it is sufficient to look only at primitive characters. Clearly if $|t - s| \geq \frac{(\log q)^4}{q}$ then the equation becomes

$$\mathbb{E}|f_\chi(t) - f_\chi(s)|^4 \leq C|t - s|^{1+\delta},$$

which, combined with Equation (17) above, proves the sequence $(\mathcal{F}_q)$ is tight for all $s, t \in [0,1]$.

Lemma 4.2 is similar to a result of Bober and Goldmakher [3, Lemma 4.1] and we use parts of their work in the proof. Unlike Section 3.3, we will consider the odd and even case at the same time.

Proof of Lemma 4.2. Using the Fourier expansion of f_χ, the difference $(f_\chi(t) - f_\chi(s))$ can be written as

$$\frac{\tau(\chi)}{2\pi i \sqrt{q}} \sum_{1 \leq |n| \leq q} \frac{\overline{\chi}(n)}{n} e(-sn)(1 - e(-(t-s)n)) + O\left(\frac{\log q}{\sqrt{q}}\right).$$

Consequently,

$$\left|f_\chi(t) - f_\chi(s)\right|^4 \leq \frac{2^4}{\pi^4} \left|\sum_{1 \leq n \leq q} \frac{\overline{\chi}(n)}{n} e(-sn)(1 - e(-(t-s)n))\right|^4 + O\left(\frac{(3 + \log q)^4}{q^2}\right).$$

Similar to Section 3.1 and [3, Lemma 4.1], we define

$$b(n) = \begin{cases} \frac{1}{n} \sum_{\substack{n_1 n_2 = n \\ n_i \leq q}} \prod_{j=1}^{2} (e(-sn_j)(1 - e(-(t-s)n_j))) & (n, q) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$\left|\sum_{1 \leq n \leq q} \frac{\overline{\chi}(n)}{n} e(-sn)(1 - e(-(t-s)n))\right|^4 = \left|\sum_{1 \leq n \leq q^2} \overline{\chi}(n) b(n)\right|^2.$$

The sum $b(n)$ can be bounded using $(1 - e(x)) \leq \min\{2, x\}$, so

(18) $$|b(n)| \leq d(n) \min\left\{\frac{2^2}{n}, (2\pi(t-s))^2\right\}.$$

As a result, taking $n = a + mq$,

(19)

$$\mathbb{E}\left|\sum_{1 \leq n \leq q^2} \overline{\chi}(n) b(n)\right|^2 = \sum_{a=1}^{q}\left|\sum_{m=0}^{q} b(a+mq)\right|^2 \leq 4\sum_{a=1}^{q} |b(a)|^2 + 4\sum_{a=1}^{q}\left|2^2 \sum_{m=1}^{q} \frac{d(a+mq)}{a+mq}\right|^2.$$

We are interested in bounding the latter inner sum,

$$\mathcal{D}_a := \sum_{m=1}^{q} \frac{d(a+mq)}{a+mq} = \sum_{\substack{q < m \leq (a+q^2) \\ m \equiv a(q)}} \frac{d(m)}{m}.$$

By Abel summation this is

$$\frac{1}{a+q^2} \sum_{\substack{q < m \leq (a+q^2) \\ m \equiv a(q)}} d(m) + \int_q^{(a+q^2)} \frac{1}{t^2} \sum_{\substack{m \leq t \\ m \equiv a(q)}} d(a) dt.$$

In order to further bound the sum, we use the Shiu's upper bound [17, Theorem 1],

$$\sum_{\substack{n \leq x \\ n \equiv a(q)}} d(n) \ll_\delta \frac{x \phi(q) \log x}{q^2} < x \cdot \frac{\log x}{q},$$

which is valid for all $x \geq q^{1+\delta}$ for any $\delta > 0$. Therefore,

$$\mathcal{D}_a = \frac{1}{a+q^2} \sum_{\substack{q \leq m \leq (a+q^2) \\ m \equiv a(q)}} d(m) + \int_q^{(a+q^2)} \frac{1}{t^2} \sum_{\substack{m \leq t \\ m \equiv a(q)}} d(a) dt$$

$$= O\left(\frac{\log(a+q^2)}{q}\right) + O\left(\int_{q^{1+\delta}}^{(a+q^2)} \frac{\log t}{qt} dt\right) + \int_q^{q^{1+\delta}} \frac{1}{t^2} \sum_{\substack{m \leq t \\ m \equiv a(q)}} d(a) dt.$$

The latter integral is bounded by the following results [6, Equation 27.11.2][7]:

$$\sum_{m \leq t} d(m) = t \log t + (2\gamma - 1)t + O(\sqrt{t}) = O(t \log t),$$

$$\sum_{\substack{m \leq t \\ m \equiv a(q)}} d(a) = \frac{1}{\phi(q)} \sum_{\substack{m \leq t \\ (m,q)=1}} d(m) + O\left((q^{1/2} + t^{1/3})t^{\epsilon}\right) = O\left(\frac{t \log t}{q}\right).$$

Combining these equations, we have

$$\mathcal{D}_a = O\left(\frac{\log q}{q}\right) + O_\delta\left(\frac{(\log q)^2}{q}\right) + O_\delta\left(\frac{(\log q)^2}{q}\right).$$

As a result, fixing $\delta > 0$, Equation (19) becomes

$$\sum_{a=1}^{q} \left| \sum_{m=0}^{q} b(a+mq) \right|^2 \leq 4 \sum_{a=1}^{q} |b(a)|^2 + O\left(\sum_{a=1}^{q} \left| \frac{(\log q)^2}{q} \right|^2 \right) = 4 \sum_{a=1}^{q} |b(a)|^2 + O\left(\frac{(\log q)^4}{q}\right).$$

Therefore the only sum left to evaluate is $\sum_{a \leq q} |b(a)|^2$. Using the bound from Equation (18) and splitting the cases $\frac{1}{a} > \pi^2(t-s)^2$ and $\frac{1}{a} < \pi^2(t-s)^2$, we have

$$\sum_{a=1}^{q} |b(a)|^2 \leq 2^4 \left(\pi^4 (t-s)^4 \sum_{a \leq \pi^{-2}(t-s)^{-2}} d(a)^2 + \sum_{\pi^{-2}(t-s)^{-2} < a \leq q} \frac{d(a)^2}{a^2} \right).$$

We combine the two sums by Rankin's trick. Taking $x = \pi^{-2}(t-s)^{-2}$,

$$\frac{1}{x^2} \sum_{a \leq x} d(a)^2 \leq \frac{x^{\sigma_1}}{x^2} \sum_{a=1}^{\infty} \frac{d(a)^2}{a^{\sigma_1}}, \qquad 1 < \sigma_1 < 2,$$

$$\sum_{a \geq x} \frac{d(a)^2}{a^2} \leq \frac{1}{x^{\sigma_2}} \sum_{a=1}^{\infty} \frac{d(a)^2}{a^{2-\sigma_2}}, \qquad 0 < \sigma_2 < 1.$$

Note σ_1, σ_2 are bounded so that the sums converge and tend to zero as $x \to \infty$. These sums are addressed by one of Ramanujan's identities [14, Section 1.3.1, Question 5]. For $Re(s) > 1$,

$$\sum_{n=1}^{\infty} \frac{d(n)^2}{n^s} = \frac{\zeta(s)^4}{\zeta(2s)}.$$

Therefore

$$\sum_{a=1}^{q} |b(a)|^2 \leq 2^4 \left(\frac{1}{x^{2-\sigma_1}} \frac{\zeta(\sigma_1)^4}{\zeta(2\sigma_1)} + \frac{1}{x^{\sigma_2}} \frac{\zeta(2-\sigma_2)^4}{\zeta(2(2-\sigma_2))} \right).$$

Taking $\sigma := \min(2 - \sigma_1, \sigma_2) \in (0,1)$ and substituting back $\pi^{-2}(t-s)^{-2} = x$,

$$\frac{1}{x^{2-\sigma_1}} \frac{\zeta(\sigma_1)^4}{\zeta(2\sigma_1)} + \frac{1}{x^{\sigma_2}} \frac{\zeta(2-\sigma_2)^4}{\zeta(2(2-\sigma_2))} \leq \frac{C}{x^{\sigma}} = C\pi^{2\sigma}(t-s)^{2\sigma},$$

for some $C = C(\sigma) > 0$. As a result,

$$\mathbb{E}\left|f_\chi(t) - f_\chi(s)\right|^4 \leq C(t-s)^{2\sigma} + O\left(\frac{(\log q)^4}{q}\right).$$

Taking $\sigma = 1 - \epsilon$ and therefore $2\sigma = 2 - \epsilon$, we have completed the proof. $\square$

Lemma 4.2 shows that the Kolmorogorov's tightness criterion argument holds for

$$|t - s|^{1+\delta_2} \gg \frac{(\log q)^4}{q},$$

where we take $\alpha = 4$ from Proposition 4.1. Therefore, combining with Equation (17), we have shown for constant K

$$\mathbb{E}\left|f_\chi(t) - f_\chi(s)\right|^4 \leq \begin{cases} |t-s|^{1+\delta_1}, & |t-s| \leq q^{-(1-\epsilon_1)} \\ K|t-s|^{1+\delta_2}, & |t-s| \geq (\log q)^4 q^{-1} \end{cases},$$

where we can choose δ_1 and δ_2 in such a way that for $\delta_1 = \frac{1-3\epsilon_1}{1-\epsilon_1}$ for $\epsilon_1 \in (0, \frac{1}{3})$ and and $\delta_2 = 1 - \epsilon_2$ where $\epsilon_2 \in (0, 1)$. This is possible as our initial parameter choices are flexible enough to allow this.

For large enough q, we have

$$\frac{(\log q)^4}{q} < \frac{1}{q^{1-\epsilon_1}}.$$

Therefore taking $\delta := \min(\delta_1, \delta_2)$, Kolmorogorov's tightness criterion holds for all t, s and $(\mathcal{F}_q)$ is tight. As a result, $(\mathcal{F}_q)_{q \text{ prime}}$ converges in distribution to the random process F as $q \to \infty$, proving Theorem 1.1. This concurs with the result from Bober, Goldmakher, Granville and Koukoulopoulos for their distribution function

$$\Phi_q(\tau) := \frac{1}{\phi(q)} \# \left\{ \chi \mod q : \max_t |S_\chi(t)| > \frac{e^\gamma}{\pi} \tau \right\}$$

weakly converging to their limiting function [2, Theorem 1.4]

$$\Phi(\tau) := \mathbb{P}\left(\max_t |F(t)| > 2e^\gamma \tau \right).$$

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A. Hussain, Department of Mathematics, University of Bristol, United Kingdom
E-mail address, ayesha.hussain@bristol.ac.uk